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Free Field Realization of WBC_n and WG_2 algebras

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Abstract

We study the BRST-cohomology in the quantum hamiltonian reduction of affine Lie algebras of non-simply laced type. We obtain the free field realization of the $W\mathbf{g}$-algebra for $\mathbf{g} = B_2$, B_3, C_3 and G_2. The WC_3 algebra is shown to be equal to the WB_3 algebra at the quantum level by duality transformation.

W-algebra symmetry [1, 2] plays an important role in the classification of rational conformal field theory and integrable systems such as Toda field theory and 2d gravity. The classical $W\mathbf{g}$-algebra associated with a simple Lie algebra $\mathbf{g}$ can be realized as the second hamiltonian structure of the generalized KdV hierarchy [3]. The classical hamiltonian reduction of affine Lie algebra provides a systematic method to the construction of generalized KdV hierarchy associated with any Lie algebra[4]. In order to study the quantum W algebra it is necessary to calculate operator product expansions of normal-ordered composite operators. The most general form of the W-algebra can be determined by consistency conditions such as the Jacobi identity. Such an approach has been done systematically for the W-algebras with two and three generators [5, 6]. However, it is technically difficult to extend this approach to general W-algebra because of the lack of Lie algebraic viewpoint.

The free field realization of the $W\mathbf{g}$ algebra is a crucial step to understand the representation of the algebra, correlation functions through screening operators, which is defined as the commutant of the W-algebra. So far the free field realization is well-understood for the $W\mathbf{g}$-algebra for simply laced Lie algebras $\mathbf{g} = A_n$ and D_n[7]. The $W\mathbf{g}$-algebra based on a simply laced Lie algebra has self-dual property in the torus, on which free bosons are compactified. Self-duality allows us to generalize the classical Miura transformation to the quantum one by taking normal ordered product of the scalar Lax operator and replacing the level k of an affine Lie algebra by a parameter $\alpha_0 = a - 1/a$, where $a = \sqrt{k + h^\vee}$ and $h^\vee$ is the dual Coxeter number.

Concerning the quantum $W\mathbf{g}$ algebra associated with a non-simply laced Lie algebra $\mathbf{g}$, it gets non-trivial quantum correction due to the lack of self-duality. In fact, the classical Miura transformation for non-simply laced Lie algebra does not work in the quantum case. The purpose of the present article is to study the free field realization of the quantum W-algebra associated with non-simply laced Lie algebras. We will use the method of quantum hamiltonian reduction [8, 9, 10] based on the BRST quantization. We examine the BRST cohomology explicitly for non-simply laced Lie algebras B_2, B_3, C_3 and G_2. An interesting properties in non-simply laced Lie algebra is the duality relation between Lie algebras B_n and C_n. This duality predicts a unique quantum W algebra for BC type Lie algebra[11]. We will confirm this duality for BC_3 case.

Let $\mathbf{g}$ be a simple Lie algebra $\mathbf{g}$ with rank r. Δ the set of roots $\mathbf{g}$, Δ_+ the set of positive roots and $\alpha_1, \ldots, \alpha_r$ the simple roots. Let $\{t_a\}$ be a basis of $\mathbf{g}$, satisfying commutation relations $[t_a, t_b] = f^c_{a,b} t_c$, where $f^c_{a,b}$ are the structure constants. In the following we shall use the Chevalley canonical basis: $\{t_a\} = \{e_\alpha, h_i\}$ $(\alpha \in \Delta, i = 1, \ldots, r)$, with commutation relations:

$$
\begin{aligned}
[e_\alpha, e_\beta] &= \begin{cases} N_{\alpha,\beta} e_{\alpha+\beta}, & \text{for } \alpha+\beta \in \Delta \\ \frac{2\alpha\cdot h}{\alpha^2}, & \text{for } \alpha+\beta = 0 \end{cases} \\
[h_i, e_\alpha] &= \alpha^i e_\alpha
\end{aligned}
\tag{1}
$$

The metric $g_{a,b}$ is defined by $g_{\alpha,\beta} = 2\delta_{\alpha+\beta,0}/\alpha^2$ and $g_{i,j} = \delta_{i,j}$, which are used to raise and lower indices of the generators. Let $\{x_0, e_0, f_0\}$ $([e_0, f_0] = x_0, [x_0, e_0] = e_0, [x_0, f_0] = -f_0)$ be a $sl(2)$ subalgebra of $\mathbf{g}$. The principal embedding of $sl(2)$ into $\mathbf{g}$ is characterized by $e_0 = \sum_{i=1}^r e_{\alpha_i}$[12]. With respect to the principal $sl(2)$ embedding $\mathbf{g}$ may be decomposed as $\mathbf{g} = \oplus_k \mathbf{g}_k$, where $\mathbf{g}_k \equiv \{x \in \mathbf{g}; (\mathrm{ad} x_0)x = kx\}$ is a $2k+1$ dimensional subspace in $\mathbf{g}$ and k runs over the exponents of $\mathbf{g}$.

The affine Lie algebra $\hat{\mathbf{g}}$ at level k is generated by the currents $\{J_a(z)\} = \{J_\alpha(z), H_i(z)\}$ $(\alpha \in \Delta, i = 1, \ldots, r)$ satisfying the operator product expansions:

$$
J_a(z) J_b(w) = \frac{k g_{ab}}{(z-w)^2} + \frac{f^c_{ab} J_c(w)}{z-w} + \cdots. \tag{2}
$$

$\hat{\mathbf{g}}$ admits a triangular decomposition $\hat{\mathbf{g}} = \hat{\mathbf{n}}_+ \oplus \hat{\mathbf{h}} \oplus \hat{\mathbf{n}}_-$, where the algebra $\hat{\mathbf{n}}_+$ ($\hat{\mathbf{n}}_-$) is generated by the currents which correspond to the positive (negative) roots of $\mathbf{g}$, $\hat{\mathbf{h}}$ by the Cartan currents $H_i(z)$.

Let us consider the constraints for the currents associated with the principal $sl(2)$ embedding into $\mathbf{g}$:

$$
\chi(J^\alpha(z)) = \begin{cases} 1 & \text{for } \alpha = \alpha_i, \;\; i = 1, \ldots, r \\ 0 & \text{for } \alpha \in \Delta_+\text{: non-simple roots.} \end{cases} \tag{3}
$$

Denote $\mathcal{M}$ the phase space with the above constraint. We may consider the reduced phase space $\mathcal{R}$ by taking quotient space with respect to the residual gauge symmetry generated by $\hat{\mathbf{n}}_+$. Classically, $\mathcal{R}$ carries the Poisson bracket structure. By choosing a special gauge, we have the representation of the classical W-algebra[4].

We shall use the BRST formalism to quantize this system. We introduce fermionic ghost fields $(b^\alpha(z), c_\alpha(z))$ $(\alpha \in \Delta_+)$ with operator product expansion: $b^\alpha(z) c_{\alpha'}(w) =$

$\delta^{\alpha}_{\alpha'}/(z-w)+\cdots$. Define the BRST charge:

$$Q_{BRST}=\int\frac{dz}{2\pi i}J_{BRST}(z), \tag{4}$$

where $J_{BRST}(z)$ is the BRST current

$$J_{BRST}(z)=\sum_{\alpha\in\Delta_+}c_\alpha(J^\alpha-\chi(J^\alpha))(z)-\frac{1}{2}\sum_{\alpha,\beta,\gamma\in\Delta_+}f^{\alpha,\beta}_{\gamma}(c_\alpha(c_\beta b^\gamma))(z). \tag{5}$$

The normal ordered product $(AB)(z)$ for operators $A(z)$ and $B(z)$ is defined by

$$(AB)(z)=\int_z\frac{dw}{2\pi i}\frac{A(w)B(z)}{w-z}. \tag{6}$$

The BRST operator Q_{BRST} satisfies the nilpotency condition $Q^2_{BRST}=0$. The physical Hilbert space may be characterized by investigating the Q_{BRST}-cohomology.

The information of the W-algebra is contained in the cohomology on the space of the universal enveloping algebra $U(\hat{\mathbf{g}})$ and the Clifford algebra $Cl_{b,c}$ generated by the fermionic ghost fields. We decompose the BRST current into two parts: $J_{BRST}(z)=J_0(z)+J_1(z)$, where

$$\begin{aligned} J_0(z) &= \sum_{\alpha\in\Delta_+}c_\alpha J^\alpha(z)-\frac{1}{2}\sum_{\alpha,\beta,\gamma\in\Delta_+}f^{\alpha,\beta}_{\gamma}(c_\alpha(c_\beta b^\gamma))(z),\\ J_1(z) &= -\sum_{\alpha\in\Delta_+}c_\alpha\chi(J^\alpha)(z). \end{aligned} \tag{7}$$

Define fermionic charges Q_i by the contour integration of $J_i(z)$. Since these charges obey the relations $(Q_0)^2=(Q_1)^2=\{Q_1,Q_0\}=0$, we may use the spectral sequence technique for the double complex generated by Q_0 and Q_1[13]. The operator Q_0 is the canonical coboundary operator of the nilpotent subalgebra $\hat{\mathbf{n}}_+$. The operator Q_1 defines a gradation in the BRST complex associated with the principal embedding $sl(2)$ into $\hat{\mathbf{g}}$. Feigin and Frenkel analyzed this complex by taking Q_0-cohomology first[9]. Under the Q_0-cohomology, the Q_1-cohomology reduces to the problem of finding conserved currents in quantum Toda field theory [15]. On the other hand, De Boer and Tjin studied the BRST-complex by taking the Q_1-cohomology first[10]. They observed that the components of non-trivial cohomology which have zero gradation in total degree of the double complex form a closed algebra and the generators are nothing but a free field realization.

For the analysis of the BRST cohomology, it is convenient to introduce new currents

$$\hat{J}^a(z)=J^a(z)+f^{a\beta}_{\gamma}(b^\gamma c_\beta)(z) \tag{8}$$

modified by ghost fields. We decompose the set of modified currents $\{\hat{J}^a\}$ into the submodules $\oplus_k V_k$ under the principal $sl(2)$ embedding, where k belongs to the set of exponents of $\mathbf{g}$. V_k denotes the set of currents $\{\hat{I}^k_m\}_{m=-k,-k+1,\ldots,k-1,k}$.

Since $Q_{BRST}(b^\alpha) = \hat{J}^\alpha - \chi(J^\alpha)$ and b^α is a non-trivial element in the BRST complex, the pairs $(b^\alpha, \hat{J}^\alpha - \chi(J^\alpha))$ form the BRST doublets and decouples from the non-trivial cohomology. Therefore we consider the BRST-cohomology on the reduced complex $\mathcal{A}_{red}$, which is spanned by other modified currents and ghost fields. First we study the Q_1-cohomology $H_{Q_1}(\mathcal{A}_{red})$ on the reduced complex $\mathcal{A}_{red}$. Since the lowest component $\hat{I}^k_{-k}$ belongs to the kernel of Q_1, we can choose $\hat{I}^k_{-k}$ as the basis of $H_{Q_1}(\mathcal{A}_{red})$. We put $\mathcal{W}^{(0)}_{k+1} = \hat{I}^k_{-k}$. In order to obtain observables of the full BRST-cohomology, we need to solve the descent equation:

$$Q_0(\mathcal{W}^{(n)}_{k+1}) = Q_1(\mathcal{W}^{(n+1)}_{k+1}). \tag{9}$$

The generators of the total BRST cohomology are given by

$$\mathcal{W}_{k+1} = \mathcal{W}^{(0)}_{k+1} - \mathcal{W}^{(1)}_{k+1} + \mathcal{W}^{(2)}_{k+1} + \cdots + (-1)^k \mathcal{W}^{(k)}_{k+1}, \tag{10}$$

where $Q_0(\mathcal{W}^{(k)}_{k+1}) = 0$.

Now we consider the BRST complex for $B^{(1)}_n$ case. The positive roots system of Lie algebra B_n is $e_i - e_j$, $e_i + e_j$ $(i < j)$ and e_i, where we introduced an orthonormal basis e_i $(i = 1, \ldots, n)$ satisfying $e_i \cdot e_j = \delta_{i,j}$. The fundamental representation of B_n is given by $(2n+1) \times (2n+1)$ matrix of the form

$$\begin{aligned} e_{e_i - e_j} &= E_{i,j} - E_{2n+2-j,2n+2-i}, \\ e_{e_i + e_j} &= E_{i,2n+2-j} - E_{j,2n+2-i}, \\ e_{e_i} &= \sqrt{2}(E_{i,n+1} - E_{n+1,2n+2-i}) \end{aligned} \tag{11}$$

and $e_{-\alpha} = {}^t e_\alpha$, $2\alpha \cdot h/\alpha^2 = [e_\alpha, e_{-\alpha}]$. The structure constants can be easily calculated from this representation.

We discuss the B_2 case for simplicity. The principal $sl(2)$ decomposition $V_1 \oplus V_3$ of $B^{(1)}_2$ is given by

$$\begin{aligned} \hat{I}^1_1 &= \hat{J}^{e_2} + \hat{J}^{e_1 - e_2}, \quad \hat{I}^1_0 = (2e_1 + e_2) \cdot \hat{H}, \quad \hat{I}^1_{-1} = 3\hat{J}^{-e_2} + 4\hat{J}^{-(e_1 - e_2)} \\ \hat{I}^3_{\pm 3} &= \hat{J}^{\pm(e_1 + e_2)}, \quad \hat{I}^3_{\pm 2} = \hat{J}^{\pm e_1}, \\ \hat{I}^3_1 &= 2\hat{J}^{e_2} - 3\hat{J}^{e_1 - e_2}, \quad \hat{I}^3_{-1} = \hat{J}^{-e_2} - 2\hat{J}^{-(e_1 - e_2)}, \quad \hat{I}^3_0 = (e_1 - 2e_2) \cdot \hat{H}. \end{aligned} \tag{12}$$

The Q_1-cohomology is generated by $\mathcal{W}_2^{(0)} = \hat{I}^1_{-1}$ and $\mathcal{W}_4^{(0)} = \hat{I}^3_{-3}$. By solving the descent equation (9), we get

$$
\begin{aligned}
\mathcal{W}_2 &= \mathcal{W}_2^{(0)} - \mathcal{W}_2^{(1)}, \\
\mathcal{W}_4 &= \mathcal{W}_4^{(0)} - \mathcal{W}_4^{(1)} + \mathcal{W}_4^{(2)} - \mathcal{W}_4^{(3)},
\end{aligned} \tag{13}
$$

where

$$
\begin{aligned}
\mathcal{W}_2^{(1)} &= \frac{1}{20}\Big((23+10k)\partial\hat{I}^1_0 + \partial\hat{I}^3_0 + 5(\hat{I}^1_0\hat{I}^1_0) + 5(\hat{I}^3_0\hat{I}^3_0)\Big), \\
\mathcal{W}_4^{(1)} &= -(\frac{11}{5}+k)\partial\hat{I}^3_{-2} + 3(\hat{I}^1_0\hat{I}^3_{-2}) - (\hat{I}^3_0\hat{I}^3_{-2}) + (\hat{I}^3_{-1}\hat{I}^3_{-1}) - 6(\hat{I}^3_{-1}\hat{I}^1_{-1}), \\
\mathcal{W}_4^{(2)} &= (-\frac{291}{50} - 5k - k^2)\partial^2\hat{I}^3_{-1} \\
&\quad + \frac{18}{5}(\hat{I}^1_0\partial\hat{I}^1_{-1}) - (14+5k)(\hat{I}^3_0\partial\hat{I}^1_{-1}) + (11+5k)(\hat{I}^1_0\partial\hat{I}^3_{-1}) - \frac{2}{5}(\hat{I}^3_0\partial\hat{I}^3_{-1}) \\
&\quad - \frac{24}{5}(\hat{I}^1_{-1}\partial\hat{I}^1_0) - (\frac{83}{5}+4k)(\hat{I}^1_{-1}\partial\hat{I}^3_0) + (5+2k)(\hat{I}^3_{-1}\partial\hat{I}^1_0) + (\frac{17}{5}+k)(\hat{I}^3_{-1}\partial\hat{I}^3_0) \\
&\quad + 18(\hat{I}^1_0(\hat{I}^3_0\hat{I}^1_{-1})) - (\hat{I}^3_0(\hat{I}^3_0\hat{I}^1_{-1})) - 6(\hat{I}^1_0(\hat{I}^1_0\hat{I}^3_{-1})) - (\hat{I}^1_0(\hat{I}^3_0\hat{I}^3_{-1})) + (\hat{I}^3_0(\hat{I}^3_0\hat{I}^3_{-1})), \\
\mathcal{W}_4^{(3)} &= (63+85k+25k^2)\partial^3\hat{I}^1_0 - (\frac{714}{125} + \frac{917k}{120} + \frac{407k^2}{120} + \frac{k^3}{2})\partial^3\hat{I}^3_0 \\
&\quad + \frac{3(13+5k)}{50}(\hat{I}^1_0\partial^2\hat{I}^1_0) + \frac{3(20+18k+4k^2)}{4}(\hat{I}^1_0\partial^2\hat{I}^3_0) + \frac{20+18k+4k^2}{8}(\hat{I}^3_0\partial^2\hat{I}^1_0) \\
&\quad + (\frac{82}{25} + \frac{51k}{20} + \frac{k^2}{2})(\hat{I}^3_0\partial^2\hat{I}^3_0) + (\frac{1091}{100} + \frac{46k}{5} + 2k^2)(\partial\hat{I}^1_0\partial\hat{I}^3_0) \\
&\quad + (\frac{281}{50} + \frac{415k}{100} + \frac{150k^2}{200})(\partial\hat{I}^3_0\partial\hat{I}^3_0) - (\frac{69}{25} + \frac{6k}{5})(\partial\hat{I}^1_0\partial\hat{I}^1_0) \\
&\quad - (9+3k)(\hat{I}^1_0(\hat{I}^3_0\partial\hat{I}^3_0)) + (\frac{69}{10}+3k)(\hat{I}^3_0(\hat{I}^3_0\partial\hat{I}^3_0)) + \frac{3}{10}(\hat{I}^1_0(\hat{I}^1_0\partial\hat{I}^1_0)) \\
&\quad - (8+\frac{7k}{2})(\hat{I}^1_0(\hat{I}^3_0\partial\hat{I}^1_0)) + (\frac{1}{20} - \frac{k}{2})(\hat{I}^3_0(\hat{I}^3_0\partial\hat{I}^1_0)) - \frac{247+110k}{20}(\hat{I}^1_0(\hat{I}^1_0\partial\hat{I}^3_0)) \\
&\quad + 3(\hat{I}^1_0(\hat{I}^1_0(\hat{I}^1_0\hat{I}^3_0))) + \frac{7}{4}(\hat{I}^1_0(\hat{I}^1_0(\hat{I}^3_0\hat{I}^3_0))) - 3(\hat{I}^1_0(\hat{I}^3_0(\hat{I}^3_0\hat{I}^3_0))).
\end{aligned} \tag{14}
$$

The last components $\mathcal{W}_2^{(2)}$ and $\mathcal{W}_4^{(3)}$ in the double complex are expressed in terms of zero degree fields $\hat{I}^1_0$ and $\hat{I}^3_0$. Introduce free bosons $\varphi_i(z)$ $(i=1,2)$ by $ia\partial\varphi_j = e_j \cdot \hat{H}$ where $a = \sqrt{k+3}$ and $\varphi_i(z)\varphi_j(w) = -\log(z-w) + \cdots$. One finds that the fields

$$
\begin{aligned}
T(z) &= \frac{40}{a^2}\mathcal{W}_2^{(2)}(z), \\
W_4(z) &= -\frac{25}{a^4}\mathcal{W}_4^{(3)}(z)
\end{aligned} \tag{15}
$$

generates the WB_2 algebra [14] with the central charge

$$c = 86 - \frac{30}{a^2} - 60a^2 = \frac{-2(12+5k)(13+6k)}{3+k}. \tag{16}$$

In terms of free fields, the generators of the WB_2 algebra are expressed as

$$\begin{aligned}
T(z) &= \frac{1}{2}(p_1^2 + p_2^2) + (\frac{3}{2a} - 2a)\partial p_1 + (\frac{1}{2a} - a)\partial p_2, \\
W_4(z) &= -p_1^4 - p_2^4 + \frac{17}{4}p_1^2 p_2^2 + 2(-\frac{3}{a} + 4a)p_1^2\partial p_1 + 17(\frac{1}{4a} - \frac{a}{2})p_1^2\partial p_2 \\
&+25(\frac{1}{2a} - \frac{a}{2})p_1 p_2 \partial p_2 + (-\frac{2}{a} + 4a)p_2^2\partial p_2 + (\frac{1}{4a} - \frac{9a}{2})(\partial p_1)p_2^2 \\
&+(\frac{39}{4} - \frac{63}{20a^2} - \frac{15a^2}{2})p_1\partial^2 p_1 + (-\frac{75}{4} + \frac{25}{4a^2} + \frac{25a^2}{2})p_1\partial^2 p_2 \\
&+(-9 + \frac{31}{10a^2} + 5a^2)p_2\partial^2 p_2 + (\frac{43}{4} - \frac{59}{10a^2} - \frac{19a^2}{4})(\partial p_1)^2 \\
&+(-5 + \frac{1}{4a^2} + 9a^2)(\partial p_1)(\partial p_2) + (-3 + \frac{21}{10a^2} + a^2)(\partial p_2)^2 \\
&+(-\frac{8}{5a^3} + \frac{631}{120a} - \frac{145a}{24} + \frac{5a^3}{2})\partial^3 p_1 + (\frac{31}{20a^3} - \frac{109}{15a} + \frac{65a}{6} - 5a^3)\partial^3 p_2,
\end{aligned} \tag{17}$$

where $p_i = i\partial\varphi_i$. Note that $W_4(z)$ is a quasi-primary field. Spin four primary field $\widetilde{W}_4(z)$ is defined by

$$\widetilde{W}_4 = W_4 + \frac{7\,(75 - 113a^2)}{4\,(75 - 226a^2 + 150a^4)}(TT) + \frac{1950 - 9101a^2 + 11245a^4 - 3300a^6}{40a^2\,(75 - 226a^2 + 150a^4)}\partial^2 T. \tag{18}$$

The present free field realization agrees with that obtained in ref. [15].

By a similar procedure we may construct the free field realization of WB_3-algebra. We define the energy-momentum tensor $T(z)$ and quasi-primary fields $W_4(z)$ and $W_6(z)$ with spins 4 and 6, respectively as follows:

$$\begin{aligned}
T(z) &= -\frac{28}{a^2}\mathcal{W}_2^{(1)}(z), \\
W_4(z) &= -\frac{48}{a^4}\mathcal{W}_4^{(3)}(z), \\
W_6(z) &= \frac{32928}{235a^6}\mathcal{W}_6^{(5)}(z).
\end{aligned} \tag{19}$$

where $a = \sqrt{k+5}$. The central charge is $c = 267 - 105/a^2 - 168a^2$. As in the case of WB_2 , we introduce free bosons $p_i \equiv e_i \cdot \hat{H}/a$. The free field realization of $T(z)$ and W_4 is given by

$$T(z) = \frac{{p_1}^2}{2} + \frac{{p_2}^2}{2} + \frac{{p_3}^2}{2} + (\frac{5}{2a} - 3a)\partial p_1 + (\frac{3}{2a} - 2a)\partial p_2 + (\frac{1}{2a} - a)\partial p_3,$$

$$
\begin{aligned}
W_4(z) \;=\;& {p_1}^4-2{p_1}^2{p_2}^2+{p_2}^4-2{p_1}^2{p_3}^2-2{p_2}^2{p_3}^2+{p_3}^4 \\
&+2(\frac{5}{a}-6a){p_1}^2(\partial p_1)-2(\frac{1}{a}-2a){p_2}^2(\partial p_1)-2(\frac{1}{a}-2a){p_3}^2(\partial p_1) \\
&-2(\frac{3}{a}-4a){p_1}^2(\partial p_2)-8(\frac{1}{a}-a)p_1p_2(\partial p_2)+2(\frac{3}{a}-4a){p_2}^2(\partial p_2)+\frac{2}{a}{p_3}^2(\partial p_2) \\
&-2(\frac{1}{a}-2a){p_1}^2(\partial p_3)-2(\frac{1}{a}-2a){p_2}^2(\partial p_3)-8(\frac{1}{a}-a)p_1p_3(\partial p_3) \\
&-8(\frac{1}{a}-a)p_2p_3(\partial p_3)+2(\frac{1}{a}-2a){p_3}^2(\partial p_3)+\frac{(5+2a^2)(17-42a^2)}{7a^2}(\partial p_1)^2 \\
&+\frac{2(3-4a^2)(-1+2a^2)}{a^2}(\partial p_1)(\partial p_2)-(\frac{64}{7}-\frac{29}{7a^2}-4a^2)(\partial p_2)^2 \\
&+(8-\frac{2}{a^2}-8a^2)(\partial p_1)(\partial p_3)-(4-\frac{2}{a^2})(\partial p_2)(\partial p_3)+(\frac{48}{7}-\frac{27}{7a^2}-4a^2)(\partial p_3)^2 \\
&+(-\frac{190}{7}+\frac{78}{7a^2}+16a^2)p_1(\partial^2 p_1)+(28-\frac{12}{a^2}-16a^2)p_1\partial^2 p_2+(\frac{6}{7}-\frac{6}{7a^2})p_2\partial^2 p_2 \\
&+(12-\frac{4}{a^2}-8a^2)p_1\partial^2 p_3+(12-\frac{4}{a^2}-8a^2)p_2\partial^2 p_3+(\frac{90}{7}-\frac{34}{7a^2}-8a^2)p_3\partial^2 p_3 \\
&+(\frac{55}{7a^3}-\frac{538}{21a}+\frac{548a}{21}-8a^3)\partial^3 p_1+(-\frac{23}{7a^3}+\frac{38}{3a}-\frac{358a}{21}+8a^3)\partial^3 p_2 \\
&+(-\frac{17}{7a^3}+\frac{230}{21a}-\frac{340a}{21}+8a^3)\partial^3 p_3. \qquad (20)
\end{aligned}
$$

Note that the W-current $W_\Delta(z)$ of spin Δ can be expressed as

$$
W_\Delta(z)=\sum_{i_1,\cdots} d^\Delta(1^{i_1}2^{i_2}\cdots,1^{j_1}2^{j_2}\cdots)p_1^{i_1}(\partial p_1)^{i_2}\cdots p_2^{j_1}(\partial p_2)^{j_2}\cdots \qquad (21)
$$

in terms of free fields $p_j=i\partial\varphi_j$ $(j=1,\ldots)$. Here $i_1,i_2\ldots$ are non-negative integers and satisfy

$$
\Delta=i_1+2i_2+\cdots+j_1+2j_2+\cdots. \qquad (22)
$$

The list of non-zero coefficients of the spin 6 field $W_6(z)$ is given in Table 1.

The spin 4 and 6 primary W-currents $\widetilde{W}_4(z)$ and $\widetilde{W}_6(z)$ are defined by

$$
\begin{aligned}
\widetilde{W}_4 \;&=\; W_4-\frac{4\,(81-113a^2)}{525-1357a^2+840a^4}(TT)-\frac{2\,(2835-11973a^2+16138a^4-6888a^6)}{7a^2\,(525-1357a^2+840a^4)}\partial^2 T, \\
\widetilde{W}_6 \;&=\; W_6+b_1(\widetilde{W}_4T)+b_2(T(TT))+b_3(T\partial^2T)+b_4(\partial T\partial T)+b_5\partial^4T+b_6\partial^2\widetilde{W}_4, \qquad (23)
\end{aligned}
$$

where

$$
\begin{aligned}
b_1 \;&=\; \frac{686\;(-11+17\,a^2)}{141\;(35-97\,a^2+56\,a^4)}, \\
b_2 \;&=\; \frac{24\;(-53482275+302341760\,a^2-644454069\,a^4+615107512\,a^6-222277440\,a^8)}{235\;(-35+48\,a^2)\;(525-1357\,a^2+840\,a^4)\;(735-1937\,a^2+1176\,a^4)},
\end{aligned}
$$

$$b_3 = \frac{2}{1645\,a^2\,(-35+48\,a^2)\,(525-1357\,a^2+840\,a^4)\,(735-1937\,a^2+1176\,a^4)}$$
$$\times\Big(12915159075 - 62611596075\,a^2 + 82595313577\,a^4 + 52890458027\,a^6 - 221412298156\,a^8$$
$$+188135793216\,a^{10} - 52327860480\,a^{12}\Big),$$
$$b_4 = \frac{2}{1645\,a^2\,(-35+48\,a^2)\,(525-1357\,a^2+840\,a^4)\,(735-1937\,a^2+1176\,a^4)}$$
$$\times\Big(4981954950 - 38746440075\,a^2 + 118960035187\,a^4 - 179703825989\,a^6 + 133280698775\,a^8$$
$$-38960324928\,a^{10} + 271656000\,a^{12}\Big),$$
$$b_5 = \frac{1}{9870\,a^4\,(-35+48\,a^2)\,(525-1357\,a^2+840\,a^4)\,(735-1937\,a^2+1176\,a^4)}\Big(162002673000$$
$$-1435942360125\,a^2 + 5197128910680\,a^4 - 9680172818260\,a^6 + 9226997505872\,a^8$$
$$-2865622277231\,a^{10} - 2298232755504\,a^{12} + 2260116476352\,a^{14} - 566577607680\,a^{16}\Big),$$
$$b_6 = \frac{49\,(-2415+12328\,a^2-16177\,a^4+4704\,a^6)}{2115\,a^2\,(35-97\,a^2+56\,a^4)}. \tag{24}$$

The WB_3 algebra is consistent with the third solution of the $W_{4,6}$ algebra in Ref. [5], which was calculated from the Jacobi identities.

We have also calculated the BRST-cohomology for an affine Lie algebra $C_3^{(1)}$. A Lie algebra C_n has simple roots $\alpha_i = (e_i - e_{i+1})/\sqrt{2}$, $(i = 1, \cdots, n-1)$ and $\alpha_n = \sqrt{2}e_n$, which are equal to simple co-roots of B_n. The cohomological analysis shows that the observables exist at spin 2,4 and 6. Spin 4 and 6 quasi-primary fields $\mathcal{W}_4^{(3)}$, $\mathcal{W}_6^{(5)}$ are apparently different from those of $B_3^{(1)}$. However, if one makes primary fields from these quasi-primary fields, one finds that the W-currents are equal to those of $B_3^{(1)}$ by the duality transformation $a \to -\sqrt{2}/a$.

For an exceptional Lie algebra G_2, we may compute its structure constant by considering the seven-dimensional representation, which can be embedded into the vector representation of B_3:

$$\begin{aligned} e_{\alpha_1} &= E_{23} - E_{56}, \\ e_{\alpha_2} &= E_{12} - E_{67} + \sqrt{2}(E_{34} - E_{45}). \end{aligned} \tag{25}$$

Here $\alpha_1 = e_1 - e_2$ and $\alpha_2 = e_2 - \frac{1}{3}(e_1 + e_2 + e_3)$ are simple roots. By investigationg the BRST-cohomology, one finds that generators of the WG_2 algebra are

$$\begin{aligned} T(z) &= \frac{-28}{a^2}\mathcal{W}_2^{(1)}(z), \\ W_6(z) &= \mathcal{W}_6^{(5)}(z). \end{aligned} \tag{26}$$

By introducing free fields $\sqrt{2}p_1 = \alpha_1 \cdot \hat{H}/a$ and $\sqrt{3/2}p_2 = \lambda_2 \cdot \hat{H}/a$, where λ_i denotes the fundamental weight of G_2 and $a = \sqrt{k+4}$, the energy momentum tensor becomes

$$T = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{\sqrt{2}}(\frac{1}{a} - a)\partial p_1 + (\frac{5}{\sqrt{6}a} - 3\sqrt{\frac{3}{2}}a)\partial p_2. \tag{27}$$

The spin 6 field W_6 is shown in Table 2. We may construct primary field as

$$\widetilde{W}_6(z) = a_1 W_6 + a_2(T(TT)) + a_3(T\partial^2 T) + a_4(\partial T \partial T) + a_5 \partial^4 T \tag{28}$$

where

$$\begin{aligned}
a_1 &= (112 - 387a^2 + 336a^4)(196 - 713a^2 + 588a^4), \\
a_2 &= 32(-6278272 + 33345830a^2 - 58292949a^4 + 33559974a^6)/27, \\
a_3 &= (-18461632 - 223338388a^2 + 1362709749a^4 - 1927045029a^6 + 34038144a^8 \\
&\quad +939880368a^{10})/1323, \\
a_4 &= (156320192 - 1352237488a^2 + 4934160444a^4 - 9498512667a^6 + 9458326773a^8 \\
&\quad -3791786796a^{10})/1323, \\
a_5 &= (-1445670912 + 18242353472a^2 - 98764008700a^4 + 294933524139a^6, \\
&\quad -522573338406a^8 + 549763546437a^{10} - 319548026700a^{12} + 79796332224a^{14})/444528.
\end{aligned} \tag{29}$$

The currents $(T, \widetilde{W}_6)$ form a closed algebra, which is consistent with the previous Jacobi-identity analysis [5, 6].

In the present article, we have examined the quantum hamiltonain reduction of non-simply affine Lie algebra with rank two and three. We have explicitly constructed higher spin currents of the W-algebra in term of free bosons. Altough it is still technically difficult to generalize the present approach to arbitrary affine Lie algebra, it would be possible to obtain the free field realization of the W-currents with spin three or four. In order to study the representation of the quantum W-alagbera, we need to introduce screening operators. The unitary representation of the WBC_n algebra is particularly interesting subject since it does not seem to correspond to any coset construction [17]. We may also consider various generalization of the present quantum hamiltonian reduction (non-principal $sl(2)$ embeddings, affine Lie superalgebras etc.) [16]. We have observed that the present quantum hamiltonian reduction gives a natural generalization of the classical Drinfeld-Sokolov approach, which can be applicable to any affine Lie algebra. Hence it is natural to ask how one can introduce an spectral parameter in the quantum

hamiltonian reduction, which would give an explict and systematic method to construct the qunatum conserved currents in a massive integrable field theory[18]. These subjects will be discussed elsewhere.

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Table 1 The W_6 current of the WB_3 algebra

partition	d^6(partition)	partition	d^6(partition)
$(1^6,,)$	1	$(,1^6,)$	1
$(,,1^6)$	1	$(1^4,1^2,)$	$-667/235$
$(1^4,,1^2)$	$-667/235$	$(1^2,1^4,)$	$-667/235$
$(,1^4,1^2)$	$-667/235$	$(1^2,,1^4)$	$-667/235$
$(,1^2,1^4)$	$-667/235$	$(1^2,1^2,1^2)$	$5526/235$
$(1^42,,)$	$3\left(5-6a^2\right)/a$	$(2,1^4,)$	$(-591+1258a^2)/235a$
$(2,,1^4)$	$(-591+1258a^2)/235a$	$(1^22,1^2,)$	$2\left(-1963+2630a^2\right)/235a$
$(1^22,,1^2)$	$2\left(-1963+2630a^2\right)/235a$	$(2,1^2,1^2)$	$2\left(95-2858a^2\right)/235a$
$(1^4,2,)$	$667\left(-3+4a^2\right)/235a$	$(,1^42,)$	$3\left(3-4a^2\right)/a$
$(,2,1^4)$	$(743-76a^2)/235a$	$(1^3,12,)$	$2744\left(-1+a^2\right)/235a$
$(1,1^32,)$	$2744\left(-1+a^2\right)/235a$	$(1^2,1^22,)$	$1334\left(-3+4a^2\right)/235a$
$(1^2,2,1^2)$	$2\left(1429-4192a^2\right)/235a$	$(,1^22,1^2)$	$2\left(-629+1296a^2\right)/235a$
$(1^4,,2)$	$667\left(-1+2a^2\right)/235a$	$(,1^4,2)$	$667\left(-1+2a^2\right)/235a$
$(,,1^42)$	$3\left(1-2a^2\right)/a$	$(1^3,,12)$	$2744\left(-1+a^2\right)/235a$
$(,1^3,12)$	$2744\left(-1+a^2\right)/235a$	$(1,,1^32)$	$2744\left(-1+a^2\right)/235a$
$(,1,1^32)$	$2744\left(-1+a^2\right)/235a$	$(1^2,1^2,2)$	$5526\left(1-2a^2\right)/235a$
$(1^2,,1^22)$	$1334\left(-1+2a^2\right)/235a$	$(,1^2,1^22)$	$1334\left(-1+2a^2\right)/235a$
$(1^22^2,,)$	$(115614-244181a^2+114450a^4)/1645a^2$	$(2^2,1^2,)$	$(-33386+45409a^2-10696a^4)/1645a^2$
$(2^2,,1^2)$	$(-33386+45409a^2-10696a^4)/1645a^2$	$(1^2,2^2,)$	$(-35514+40593a^2-3752a^4)/1645a^2$
$(,1^22^2,)$	$(55862-114541a^2+44562a^4)/1645a^2$	$(,2^2,1^2)$	$(22110-36239a^2+15456a^4)/1645a^2$
$(1,12^2,)$	$2744\left(-1+a^2\right)\left(3-4a^2\right)/235a^2$	$(1^2,,2^2)$	$(1838-4767a^2+4256a^4)/1645a^2$
$(,1^2,2^2)$	$(1838-4767a^2+4256a^4)/1645a^2$	$(,,1^22^2)$	$(16382-25445a^2-5054a^4)/1645a^2$
$(1,1,2^2)$	$16464\left(-1+a^2\right)^{2}{}^{2}/235a^2$	$(1,,12^2)$	$2744\left(-1+a^2\right)\left(1-2a^2\right)/235a^2$
$(,1,12^2)$	$2744\left(-1+a^2\right)\left(1-2a^2\right)/235a^2$	$(1^22,2,)$	$2\left(1963-2630a^2\right)\left(-3+4a^2\right)/235a^2$
$(2,1^22,)$	$2\left(591-1258a^2\right)\left(-3+4a^2\right)/235a^2$	$(2,2,1^2)$	$2\left(-1087+650a^2+3200a^4\right)/235a^2$
$(12,12,)$	$4\left(-33888+61691a^2-23863a^4\right)/1645a^2$	$(1^22,,2)$	$2\left(1963-2630a^2\right)\left(-1+2a^2\right)/235a^2$
$(2,1^2,2)$	$2\left(95-2858a^2\right)\left(1-2a^2\right)/235a^2$	$(2,,1^22)$	$2\left(591-1258a^2\right)\left(-1+2a^2\right)/235a^2$
$(12,,12)$	$4\left(-33888+61691a^2-23863a^4\right)/1645a^2$	$(2,1,12)$	$2744\,(1-a)\left(1-6a^2\right)/235a^2$
$(1^2,2,2)$	$2\left(1429-4192a^2\right)\left(1-2a^2\right)/235a^2$	$(,1^22,2)$	$2\left(629-1296a^2\right)\left(-1+2a^2\right)/235a^2$
$(,2,1^22)$	$2\left(-1+2a^2\right)\left(-743+76a^2\right)/235a^2$	$(1,12,2)$	$2744\left(-1+a^2\right)\left(-1+2a^2\right)/47a^2$
$(1,2,12)$	$2744\left(-1+a^2\right)\left(-9+14a^2\right)/235a^2$	$(,12,12)$	$4\left(4528-15141a^2+14553a^4\right)/1645a^2$
$(2^3,,)$	$\left(-5+6a^2\right)\left(-17448+26103a^2-5320a^4\right)/1645a^3$	$(,2^3,)$	$\left(-3+4a^2\right)\left(-10336+14903a^2-1232a^4\right)/1645a^3$
$(,,2^3)$	$\left(-1+2a^2\right)\left(2824-8393a^2+8904a^4\right)/1645a^3$	$(2^2,2,)$	$\left(-3+4a^2\right)\left(33386-45409a^2+10696a^4\right)/1645a^3$
$(2^2,,2)$	$\left(-1+2a^2\right)\left(33386-45409a^2+10696a^4\right)/1645a^3$	$(2^2,,2)$	$\left(-1+2a^2\right)\left(33386-45409a^2+10696a^4\right)/1645a^3$
$(2,2^2,)$	$(-119946+358425a^2-300734a^4+60928a^6)/1645a^3$	$(,2^2,2)$	$\left(-1+2a^2\right)\left(-22110+36239a^2-15456a^4\right)/1645a^3$
$(2,,2^2)$	$(-86850+214841a^2-142198a^4+12880a^6)/1645a^3$	$(,2,2^2)$	$(24722-117693a^2+185500a^4-93856a^6)/1645a^3$
$(2,2,2)$	$2\left(1-2a^2\right)\left(-1087+650a^2+3200a^4\right)/235a^3$	$(1^33,,)$	$(33947-71197a^2+42140a^4)/1645a^2$
$(13,1^2,)$	$(-12701+44051a^2-34692a^4)/1645a^2$	$(13,,1^2)$	$(-12701+44051a^2-34692a^4)/1645a^2$
$(1^3,3,)$	$1372\left(-1+a^2\right)\left(3-4a^2\right)/235a^2$	$(,1^33,)$	$(5135-3969a^2+3724a^4)/1645a^2$
$(1^2,13,)$	$(-41513+111279a^2-73108a^4)/1645a^2$	$(1,1^23,)$	$1372\left(-1+a^2\right)\left(3-4a^2\right)/235a^2$
$(1,3,1^2)$	$1372\,(1-a)\left(3-8a^2\right)/235a^2$	$(,13,1^2)$	$(16111-61593a^2+42140a^4)/1645a^2$
$(1^3,,3)$	$1372\left(-1+a^2\right)\left(1-2a^2\right)/235a^2$	$(,1^3,3)$	$1372\left(-1+a^2\right)\left(1-2a^2\right)/235a^2$
$(,,1^33)$	$(-4469+24843a^2-15484a^4)/1645a^2$	$(1^2,1,3)$	$1372\left(-1+a^2\right)\left(-1+2a^2\right)/47a^2$
$(1^2,,13)$	$3\left(2169-10927a^2+7644a^4\right)/1645a^2$	$(1,1^2,3)$	$1372\left(-1+a^2\right)\left(-1+2a^2\right)/47a^2$
$(,1^2,13)$	$3\left(2169-10927a^2+7644a^4\right)/1645a^2$	$(1,,1^23)$	$1372\left(-1+a^2\right)\left(1-2a^2\right)/235a^2$
$(,1,1^23)$	$1372\left(-1+a^2\right)\left(1-2a^2\right)/235a^2$	$(123,,)$	$\left(-5+6a^2\right)\left(-49863+91875a^2-32830a^4\right)/1645a^3$
$(13,2,)$	$\left(-3+4a^2\right)\left(12701-44051a^2+34692a^4\right)/1645a^3$	$(3,12,)$	$2\left(-37042+110473a^2-105301a^4+27930a^6\right)/1645a^3$
$(3,,12)$	$2\left(-37042+110473a^2-105301a^4+27930a^6\right)/1645a^3$	$(13,,2)$	$\left(-1+2a^2\right)\left(12701-44051a^2+34692a^4\right)/1645a^3$
$(3,,12)$	$2\left(-37042+110473a^2-105301a^4+27930a^6\right)/1645a^3$	$(12,3,)$	$2\left(-3+4a^2\right)\left(33888-61691a^2+23863a^4\right)/1645a^3$
$(2,13,)$	$(-92317+267649a^2-264894a^4+92904a^6)/1645a^3$	$(1,23,)$	$196\left(-1+a^2\right)\left(81-112a^2+28a^4\right)/235a^3$
$(,123,)$	$\left(-3+4a^2\right)\left(-21051+24647a^2+5586a^4\right)/1645a^3$	$(,3,12)$	$2\left(40338-157517a^2+190659a^4-77420a^6\right)/1645a^3$
$(1,3,2)$	$1372\,(-1-a)\,(1-a)\left(3-8a^2\right)\left(-1+2a^2\right)/235a^3$	$(,13,2)$	$\left(-1+2a^2\right)\left(-16111+61593a^2-42140a^4\right)/1645a^3$

$(,3,12)$	$2\left(40338-157517\,a^2+190659\,a^4-77420\,a^6\right)/1645\,a^3$	$(12,,3)$	$2\left(-1+2\,a^2\right)\left(33888-61691\,a^2+23863\,a^4\right)/1645\,a^3$
$(2,1,3)$	$1372\,(-1-a)\,(1-a)\left(1-6\,a^2\right)\left(-1+2\,a^2\right)/235\,a^3$	$(2,,13)$	$(-82713+181213\,a^2-72814\,a^4-22344\,a^6)/1645\,a^3$
$(1,2,3)$	$1372\,(-1-a)\,(1-a)\left(9-14\,a^2\right)\left(-1+2\,a^2\right)/235\,a^3$	$(,12,3)$	$2\left(-1+2a^2\right)\left(-4528+15141\,a^2-14553\,a^4\right)/1645\,a^3$
$(,,123)$	$\left(1-2\,a^2\right)\left(11447+4165\,a^2-24794\,a^4\right)/1645\,a^3$	$(1,,23)$	$196\,(-1-a)\,(1-a)\left(-17+70\,a^2-56\,a^4\right)/235\,a^3$
$(,1,23)$	$196\,(-1-a)\,(1-a)\left(-17+70\,a^2-56\,a^4\right)/235\,a^3$	$(,,123)$	$\left(1-2\,a^2\right)\left(11447+4165\,a^2-24794\,a^4\right)/1645\,a^3$

$(3^2,,)$	$(164846-664866\,a^2+946483\,a^4-551593\,a^6+106820\,a^8/3290\,a^4)$
$(,3^2,)$	$(26542-141610\,a^2+217943\,a^4-109025\,a^6+7840\,a^8/3290\,a^4)$
$(,,3^2)$	$(-26146+31662\,a^2+87363\,a^4-154889\,a^6+63700\,a^8)/3290\,a^4$
$(3,3,)$	$\left(-3+4\,a^2\right)\left(37042-110473\,a^2+105301\,a^4-27930\,a^6\right)/1645\,a^4$
$(3,,3)$	$\left(1-2\,a^2\right)\left(-37042+110473\,a^2-105301\,a^4+27930\,a^6\right)/1645\,a^4$
$(,3,3)$	$\left(1-2\,a^2\right)\left(40338-157517\,a^2+190659\,a^4-77420\,a^6\right)/1645\,a^4$
$(1^24,,)$	$\left(-5+6\,a^2\right)\left(-44217+69797\,a^2-30380\,a^4\right)/9870\,a^3$
$(4,1^2,)$	$(-25875+118361\,a^2-140014\,a^4+48216\,a^6)/9870\,a^3$
$(4,,1^2)$	$(-25875+118361\,a^2-140014\,a^4+48216\,a^6)/9870\,a^3$
$(24,,)$	$(837297-3054953\,a^2+3900286\,a^4-1953798\,a^6+270480\,a^8)/9870\,a^4$
$(4,2,)$	$\left(3-4\,a^2\right)\left(-25875+118361\,a^2-140014\,a^4+48216\,a^6\right)/9870\,a^4$
$(4,,2)$	$\left(1-2\,a^2\right)\left(-25875+118361\,a^2-140014\,a^4+48216\,a^6\right)/9870\,a^4$
$(1^2,4,)$	$\left(-3+4\,a^2\right)\left(76519-151095\,a^2+75264\,a^4\right)/9870\,a^3$
$(,1^24,)$	$\left(3-4\,a^2\right)\left(5801+16639\,a^2-17640\,a^4\right)/9870\,a^3$
$(,4,1^2)$	$(50331-192807\,a^2+303100\,a^4-159936\,a^6)/9870\,a^3$
$(1,14,)$	$392\left(-1+a^2\right)\left(51-133\,a^2+84\,a^4\right)/705\,a^3$
$(2,4,)$	$\left(-3+4\,a^2\right)\left(267347-715181\,a^2+552986\,a^4-105840\,a^6\right)/9870\,a^4$
$(,24,)$	$(-150063+683983\,a^2-1117470\,a^4+766318\,a^6-183456\,a^8)/9870\,a^4$
$(,4,2)$	$\left(1-2\,a^2\right)\left(50331-192807\,a^2+303100\,a^4-159936\,a^6\right)/9870\,a^4$
$(1^2,,4)$	$\left(-1+2\,a^2\right)\left(-19521+89005\,a^2-68796\,a^4\right)/9870\,a^3$
$(,1^2,4)$	$\left(-1+2\,a^2\right)\left(-19521+89005\,a^2-68796\,a^4\right)/9870\,a^3$
$(,,1^24)$	$\left(-1+2\,a^2\right)\left(4469-21553\,a^2+15484\,a^4\right)/3290\,a^3$
$(1,1,4)$	$8232\left(-1+a^2\right)^{2\,2}\left(1-2\,a^2\right)/235\,a^3$
$(1,,14)$	$784\left(-1+a^2\right)\left(-6+28\,a^2-21\,a^4\right)/705\,a^3$
$(,1,14)$	$784\left(-1+a^2\right)\left(-6+28\,a^2-21\,a^4\right)/705\,a^3$
$(2,,4)$	$\left(-1+2\,a^2\right)\left(248139-551913\,a^2+236054\,a^4+67032\,a^6\right)/9870\,a^4$
$(,2,4)$	$\left(1-2\,a^2\right)\left(58563-268267\,a^2+485576\,a^4-275184\,a^6\right)/9870\,a^4$
$(,,24)$	$(-149823+429535\,a^2-330394\,a^4-93478\,a^6+143472\,a^8)/9870\,a^4$
$(15,,)$	$7\left(-1+a^2\right)\left(-612+1729\,a^2-1414\,a^4+280\,a^6\right)/235\,a^4$
$(1,5,)$	$98\left(-1+a^2\right)\left(-3+4\,a^2\right)\left(-20+35\,a^2-14\,a^4\right)/235\,a^4$
$(,15,)$	$21\left(1-a^2\right)\left(-76+287\,a^2-378\,a^4+168\,a^6\right)/235\,a^4$
$(1,,5)$	$98\left(1-a^2\right)\left(1-2\,a^2\right)\left(8-35\,a^2+28\,a^4\right)/235\,a^4$
$(,1,5)$	$98\left(1-a^2\right)\left(1-2\,a^2\right)\left(8-35\,a^2+28\,a^4\right)/235\,a^4$
$(,,15)$	$7\left(-1+a^2\right)\left(116-147\,a^2-238\,a^4+280\,a^6\right)/235\,a^4$
$(6,,)$	$\left(-5+6\,a^2\right)\left(-191688+633923\,a^2-684691\,a^4+263914\,a^6-27440\,a^8\right)/98700\,a^5$
$(,6,)$	$\left(-3+4\,a^2\right)\left(280280-1119493\,a^2+1614781\,a^4-946190\,a^6+164640\,a^8\right)/98700\,a^5$
$(,,6)$	$\left(1-2\,a^2\right)\left(-170520+384101\,a^2+86499\,a^4-705698\,a^6+411600\,a^8\right)/98700\,a^5$

Table 2 The W_6 current of WG_2 algebra

partition	d^6(partition)
$\left(1^6,\right)$	$a^6\left(2-9a^2\right)\left(1-2a^2\right)\left(-6400+20103a^2-15552a^4\right)/432$
$\left(,1^6\right)$	$a^6\left(2-3a^2\right)\left(3-2a^2\right)\left(1728-6701a^2+6400a^4\right)/144$
$\left(1^4,1^2\right)$	$a^6\left(118912-810842a^2+2038215a^4-2238714a^6+905472a^8\right)/144$
$\left(1^2,1^4\right)$	$a^6\left(-33536+248746a^2-679405a^4+810842a^6-356736a^8\right)/48$
$\left(1^42,\right)$	$(-1-a)\,(1-a)\,a^5\left(2-9a^2\right)\left(1-2a^2\right)\left(6400-20103a^2+15552a^4\right)/72\sqrt{2}$
$\left(2,1^4\right)$	$(-1-a)\,(1-a)\,a^5\left(33536-248746a^2+679405a^4-810842a^6+356736a^8\right)/24\sqrt{2}$
$\left(1^32,1\right)$	$a^5\left(-2+3a^2\right)\left(-112+387a^2-336a^4\right)\left(196-713a^2+588a^4\right)/6\sqrt{6}$
$\left(12,1^3\right)$	$a^5\left(2-5a^2\right)\left(-196+713a^2-588a^4\right)\left(112-387a^2+336a^4\right)/6\sqrt{6}$

$\left(1^22,1^2\right)$	$(-1-a)(1-a)a^5\left(-118912+810842a^2-2038215a^4+2238714a^6-905472a^8\right)/36\sqrt{2}$
$\left(1^4,2\right)$	$a^5\left(67712-597154a^2+2099733a^4-3651381a^6+3117690a^8-1036800a^{10}\right)/72\sqrt{6}$
$\left(,1^42\right)$	$a^5\left(2-3a^2\right)\left(3-2a^2\right)\left(-5+9a^2\right)\left(-1728+6701a^2-6400a^4\right)/24\sqrt{6}$
$\left(1^2,1^22\right)$	$a^5\left(-79872+703202a^2-2448087a^4+4223929a^6-3619746a^8+1234944a^{10}\right)/12\sqrt{6}$
$\left(1^22^2,\right)$	$a^4\left(131680-1338586a^2+5648195a^4-12663269a^6+15903183a^8-10602090a^{10}+2931552a^{12}\right)/72$
$\left(2^2,1^2\right)$	$a^4\left(-65216+1017734a^2-5958187a^4+17418247a^6-27317505a^8+22012686a^{10}-7162560a^{12}\right)/72$
$\left(12^2,1\right)$	$(1-a)a^4(1+a)\left(2-3a^2\right)\left(112-387a^2+336a^4\right)\left(196-713a^2+588a^4\right)/6\sqrt{3}$
$\left(1^2,2^2\right)$	$a^4\left(-56640+588586a^2-2533419a^4+5784095a^6-7382049a^8+4992498a^{10}-1401408a^{12}\right)/72$
$\left(,1^22^2\right)$	$a^4\left(2-3a^2\right)\left(3-2a^2\right)\left(23376-170801a^2+467977a^4-571761a^6+263376a^8\right)/72$
$\left(1^22,2\right)$	$(1-a)a^4(1+a)\left(67712-597154a^2+2099733a^4-3651381a^6+3117690a^8-1036800a^{10}\right)/36\sqrt{3}$
$\left(2,1^22\right)$	$(1-a)a^4(1+a)\left(-79872+703202a^2-2448087a^4+4223929a^6-3619746a^8+1234944a^{10}\right)/12\sqrt{3}$
$(12,12)$	$a^4\left(-33040+359780a^2-1620064a^4+3867057a^6-5168870a^8+3672324a^{10}-1083600a^{12}\right)/6$
$\left(2^3,\right)$	$-\left(1-a^2\right)a^3\left(-137280+1400402a^2-5946623a^4+13597517a^6-17848893a^8+12848490a^{10}-3965760a^{12}\right)/108\sqrt{2}$
$\left(,2^3\right)$	$a^3\left(3-2a^2\right)\left(-2+3a^2\right)\left(-19104+170473a^2-610976a^4+1101699a^6-999225a^8+365472a^{10}\right)/108\sqrt{6}$
$\left(2^2,2\right)$	$a^3\left(-62656+909454a^2-5536793a^4+18456374a^6-36502251a^8+42874803a^{10}-27678078a^{12}+7563456a^{14}\right)/36\sqrt{6}$
$\left(2,2^2\right)$	$-\left(1-a^2\right)a^3\left(56640-588586a^2+2533419a^4-5784095a^6+7382049a^8-4992498a^{10}+1401408a^{12}\right)/36\sqrt{2}$
$\left(1^33,\right)$	$a^4\left(75040-793904a^2+3492048a^4-8161584a^6+10678311a^8-7406100a^{10}+2122848a^{12}\right)/108$
$\left(3,1^3\right)$	$\left(1-a^2\right)a^4\left(2-5a^2\right)\left(-112+387a^2-336a^4\right)\left(196-713a^2+588a^4\right)/12\sqrt{3}$
$\left(1^23,1\right)$	$\left(1-a^2\right)a^4\left(2-3a^2\right)\left(112-387a^2+336a^4\right)\left(196-713a^2+588a^4\right)/12\sqrt{3}$
$\left(13,1^2\right)$	$a^4\left(-92064+1092656a^2-5273920a^4+13291224a^6-18488667a^8+13483908a^{10}-4034016a^{12}\right)/36$
$\left(,1^33\right)$	$a^4\left(2-3a^2\right)\left(3-2a^2\right)\left(6608-50748a^2+145027a^4-182340a^6+85008a^8\right)/36$
$\left(1^2,13\right)$	$a^4\left(-48160+495864a^2-2105150a^4+4721997a^6-5906274a^8+3908664a^{10}-1070496a^{12}\right)/36$
$(123,)$	$-\left(1-a^2\right)a^3\left(-72240+750196a^2-3219428a^4+7286817a^6-9141426a^8+6002964a^{10}-1605744a^{12}\right)/18\sqrt{2}$
$(23,1)$	$5a^5\left(2-3a^2\right)\left(1-2a^2\right)\left(112-387a^2+336a^4\right)\left(196-713a^2+588a^4\right)/12\sqrt{6}$
$(13,2)$	$a^3\left(-130368+1709744a^2-9495700a^4+29004108a^6-52673817a^8+56916054a^{10}-33906168a^{12}+8600256a^{14}\right)/36\sqrt{6}$
$(3,12)$	$-\left(1-a^2\right)a^3\left(33040-359780a^2+1620064a^4-3867057a^6+5168870a^8-3672324a^{10}+1083600a^{12}\right)/6\sqrt{2}$
$(12,3)$	$a^3\left(-113120+1388088a^2-7248616a^4+20891364a^6-35940285a^8+36981270a^{10}-21118752a^{12}+5171040a^{14}\right)/36\sqrt{6}$
$(2,13)$	$-\left(1-a^2\right)a^3\left(48160-495864a^2+2105150a^4-4721997a^6+5906274a^8-3908664a^{10}+1070496a^{12}\right)/18\sqrt{2}$
$(,123)$	$a^3\left(2-3a^2\right)\left(-3+2a^2\right)\left(-10864+99324a^2-362902a^4+663211a^6-607056a^8+222768a^{10}\right)/12\sqrt{6}$
$\left(3^2,\right)$	$a^2\left(95200-718280a^2+508692a^4+12103458a^6-56831649a^8 +121031121a^{10}-140408586a^{12}+86215968a^{14}-22044960a^{16}\right)/432$
$\left(,3^2\right)$	$a^2\left(2-3a^2\right)\left(3-2a^2\right)\left(20720-223548a^2+1004512a^4-2407563a^6+3251043a^8-2349432a^{10}+710640a^{12}\right)/432$
$(3,3)$	$\left(1-a^2\right)a^2\left(-113120+1388088a^2-7248616a^4+20891364a^6-35940285a^8 +36981270a^{10}-21118752a^{12}+5171040a^{14}\right)/72\sqrt{3}$
$\left(1^24,\right)$	$-\left(1-a^2\right)a^3\left(-150080+1575008a^2-6860690a^4+15915525a^6-20792592a^8+14532264a^{10}-4245696a^{12}\right)/216\sqrt{2}$
$\left(4,1^2\right)$	$-\left(1-a^2\right)a^3\left(2-3a^2\right)\left(276192-2804224a^2+11210003a^4-22039560a^6+21287304a^8-8068032a^{10}\right)/216\sqrt{2}$
$(14,1)$	$(-1-2a)(1-2a)a^3\left(-2+3a^2\right)^2\left(112-387a^2+336a^4\right)\left(196-713a^2+588a^4\right)/36\sqrt{6}$
$(24,)$	$a^2\left(24640+13264a^2-2263626a^4+15398627a^6-49506115a^8 +90542514a^{10}-96858504a^{12}+56673864a^{14}-14043456a^{16}\right)/216$
$(4,2)$	$\left(1-a^2\right)a^2\left(2-3a^2\right)\left(-195552+2237432a^2-10588825a^4+26573058a^6-37325448a^8 +27827064a^{10}-8600256a^{12}\right)/216\sqrt{3}$
$\left(1^2,4\right)$	$a^3\left(-1+2a^2\right)\left(42560-423104a^2+1733762a^4-3754133a^6+4521420a^8-2863332a^{10}+743904a^{12}\right)/72\sqrt{6}$
$\left(,1^24\right)$	$a^3\left(2-3a^2\right)\left(-3+2a^2\right)\left(-1+2a^2\right)\left(22176-169536a^2+479893a^4-593946a^6+271152a^8\right)/72\sqrt{6}$
$(2,4)$	$\left(1-a^2\right)a^2\left(1-2a^2\right)\left(-42560+423104a^2-1733762a^4+3754133a^6-4521420a^8+2863332a^{10}-743904a^{12}\right)/72\sqrt{3}$
$(,24)$	$a^2\left(2-3a^2\right)\left(1-2a^2\right)\left(3-2a^2\right)\left(16800-151824a^2+548309a^4-990687a^6+895194a^8-323568a^{10}\right)/216$
$(15,)$	$a^2\left(2-9a^2\right)\left(2-3a^2\right)\left(1-2a^2\right)\left(-3920+32256a^2-104968a^4+168849a^6-134316a^8+42336a^{10}\right)/108$
$(5,1)$	$(1-a)a^2(1+a)\left(1-3a^2\right)\left(2-3a^2\right)\left(1-2a^2\right)\left(-112+387a^2-336a^4\right)\left(196-713a^2+588a^4\right)/36\sqrt{3}$
$(,15)$	$a^2\left(2-3a^2\right)\left(1-2a^2\right)\left(3-2a^2\right)\left(1568-14924a^2+56283a^4-104968a^6+96768a^8-35280a^{10}\right)/36$
$(6,)$	$-\left(1-a^2\right)a\left(2-3a^2\right)\left(1-2a^2\right)\left(117600-1405180a^2+6868151a^4 -17605578a^6+24993450a^8-18647496a^{10}+5715360a^{12}\right)/1620\sqrt{2}$
$(,6)$	$a\left(2-3a^2\right)\left(1-2a^2\right)\left(3-2a^2\right)\left(-5+9a^2\right)\left(-1568+14924a^2-56283a^4+104968a^6-96768a^8+35280a^{10}\right)/540\sqrt{6}$